note 16 Jan.

Show that $Re(iz) = -Im(z)$ and $Im(iz) = Re(z)$.

Slotion: Let $z = x+iy$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.
Then $iz = i(x+iy) = ix + i^2 y = -y + ix$.
Hence $Re(iz) = -y = -Im(z)$

$Im(iz) = x = Re(z)$.

Q2. Let $z = \frac{1+2i}{3-4i} + \frac{2-i}{5i}$. Find $Re(z)$ and $Im(z)$.

Solution: $z = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)i}{5i^2}$

$= \frac{3+6i+4i+8i^2}{9-16i^2} + \left(\frac{2i-i^2}{-5}\right)$

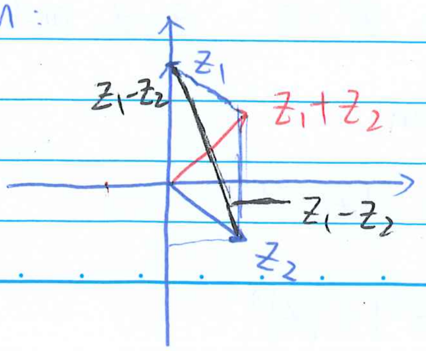
$= \frac{-5+10i}{25} - \frac{2i+1}{5}$

$= \frac{-2}{5}$

Thus $Re(z) = -\frac{2}{5}$ $Im(z) = 0$.

Q3. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$, where $z_1 = 2i$, $z_2 = 1-i$

Solution:



Q4. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Show

$$|(x_1+iy_1)(x_2+iy_2)| = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)},$$

and $|z^n| = |z|^n$.

Solution:

Note that
$$
\begin{aligned}
|z_1 z_2| &= |(x_1+iy_1)(x_2+iy_2)| \\
&= |x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)| \\
&= \sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} \\
&= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}.
\end{aligned}
$$

Since $|z_1| = \sqrt{x_1^2+y_1^2}$ $\quad |z_2| = \sqrt{x_2^2+y_2^2}$

Then $|z_1 z_2| = |z_1|\,|z_2|$.

Next we apply mathematic induction to show $|z^n| = |z|^n$.

For $n=1$, $|z| = |z|$. we assume $|z^k| = |z|^k$, $k \in \mathbb{Z}^+$

Considering $|z^{k+1}| = |z^k z|$. Let $z^k = z_1$, $z = z_2$ and applying $|z_1 z_2| = |z_1|\,|z_2|$, we obtain

$$|z^k z| = |z^k|\,|z| = |z|^k\,|z| = |z|^{k+1}.$$

By mathematic induction, we have $|z^n| = |z|^n$.

Q5. Compute $\overline{z+3i}$ and $\overline{(2+i)^2}$.

Solution

$$\overline{z+3i} = \bar{z} - 3i = z - 3i$$

$$\overline{(2+i)^2} = (2-i)^2 = 4+i^2 - 4i = 3-4i$$

Q6. Sketch the set of points determined by $Re(\bar{z}-i) = 2$.

Solution:

Since $Re(\bar{z}-i) = \dfrac{(\bar{z}-i) + \overline{(\bar{z}-i)}}{2} = \dfrac{\bar{z}-i+z+i}{2} = \dfrac{\bar{z}+z}{2}$

then we have $Re(\bar{z}-i) = Re(z) = 2$.

Q7. prove $|z_1+z_2| \le |z_1| + |z_2|$.

Solution:

$$|z_1+z_2|^2 = (z_1+z_2)\,(\overline{z_1+z_2}) = z_1\bar{z}_1 + z_2\bar{z} + z_1\bar{z}_2 + z_2\bar{z}_1$$
$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

Note that $Re(z_1\bar{z}_2) = \dfrac{z_1\bar{z}_2 + \overline{z_1\bar{z}_2}}{2} = \dfrac{z_1\bar{z}_2 + z_2\bar{z}_1}{2}.$

Then $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$.

Since

$$Re(z_1\bar{z}_2) \le |Re(z_1\bar{z}_2)| \le |z_1\bar{z}_2| = |z_1|\,|z_2|.$$

Then we have $|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| = (|z_1|+|z_2|)^2$

Therefore.

$$|z_1| + |z_2| \geq |z_1+z_2|.$$